2024 AGMC Tianyi New Year Cup Math Competition

【Individual Round - Senior Group】

Question Sheet

(Full Marks: 100 points   Duration: 4 hours)

Read the following instructions carefully before you start the exam.

I. The competition consists of 14 questions with a total score of 100 points and a duration of 4 hours. It includes two sessions: the first session from 10:00 AM to 11:20 AM for Multiple-choice and Fill-in-the-blank questions; and the second session from 2:00 PM to 4:40 PM for Free-response questions.

II. Answers should be written on the answer sheet. Answers written on the question sheet or draft paper will be considered invalid.

III. This competition is an online open-book exam, and consulting paper materials is allowed. The use of electronic devices such as computers, mobile phones, calculators, etc. is prohibited. Any use of electronic devices will result in disqualification.

IV. This competition is an individual round. Consequently, any collaboration is prohibited. Any forms of collaboration will result in disqualification.

V. After the time is up, please take a photo of the answer sheet and upload it to Rainclassroom. Late submissions will be considered invalid.

Section A【Multiple Choice Questions】

(*This section contains 5 questions, each worth 4 points, for a total of 20 points*)

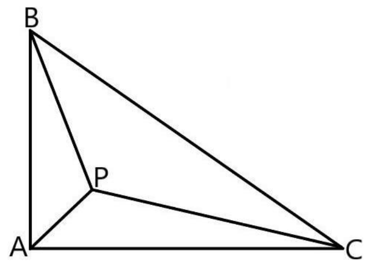
1. Let $f(x) = x^a + 2^x$, $f(10) = 2024$, the number of zeros of $f(x)$ is ( ).

A. 0 B. 1 C. 2 D. 3

2. The unit digit of $2022^{2023^{2024}}$ is ( ).

A. 2 B. 4 C. 6 D. 8

3. $\triangle ABC$ is a right triangle, where $\angle BAC$=90°. Point P is inside $\triangle BAC$. connect AP, BP, and CP. If AP=1, BP=2, CP=3, $BC_{\max}$ and $S_{\triangle ABC\max}$ are respectively ( ).



A. $1 + 2\sqrt{3}$ and $4\sqrt{2} - 1$ B. $1 + 2\sqrt{3}$ and $3 + \sqrt{3}$

C. $2\sqrt{2} + \sqrt{3}$ and $4\sqrt{2} - 1$ D. $2\sqrt{2} + \sqrt{3}$ and $3 + \sqrt{3}$

4. Rounding symbols include the ceiling symbol and the floor symbol: $\lceil x \rceil$ represents the smallest integer greater than or equal to x; $\lfloor x \rfloor$ represents the largest integer less than or equal to x. For example, we have $\lceil \mathrm{e} \rceil = \lceil 3 \rceil = \lfloor \pi \rfloor = \lfloor 3 \rfloor = 3$ according to this rule. Compute $\sum_{n=1}^{215}\left(\left\lceil \frac{7}{36}n^2 + \frac{7}{6}n + \frac{3}{4} \right\rceil - \left\lfloor \frac{7}{36}n^2 + \frac{7}{9}n + \frac{16}{9} \right\rfloor\right) =$ ( ).

A. 8820 B. 8848 C. 8898 D. 8988

5. A sequence is called a *Fibonacci Sequence* if it satisfies $a_{n+2} = a_n + a_{n+1}$, and $a_1 = a_2 = 1$. If there exists $k \in \mathbb{N}^+$ such that $a_k \equiv 0 \pmod{127}$ and $a_{k+1} \equiv 1 \pmod{127}$, then find $k_{\min} =$ ( ).

A. 2^8 B. 5×2^7 C. 2^{10} D. 5×2^9

Section B【Fill in the Blanks】

(*This section contains 5 questions, each worth 4 points, for a total of 20 points*)

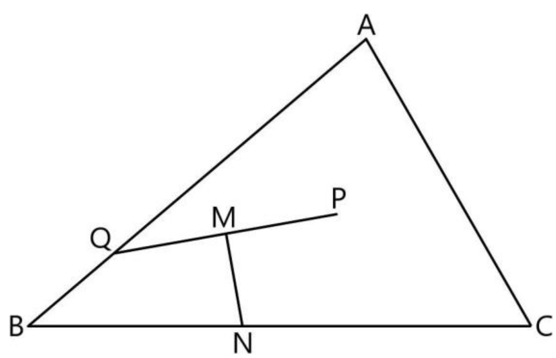
6. Given that $\theta \in [0, \frac{\pi}{2}]$, $\sin 2\theta = \cos^2 \theta$, compute $\sin 4\theta =$ __________ .

7. Let $f(x) = x^2 + ax + 2a + 1$, the equation $f(f(x)) = x$ has two real roots and two non-real roots, then the range of the real number a is __________ .

8. Let $k \in \mathbb{N}^+$. The indeterminate equation $x^2 + y^2 = k$ has 6 distinct pairs of positive integer solutions (x, y). Compute $k_{\min} =$ __________ .

9. The inner tangent ellipse of a triangle refers to the ellipse that is tangent to all three sides of the triangle. The three sides of $\triangle ABC$ are 4, 5, 6, and the points O and P are their circumcenter and orthocenter, respectively. Points O and P are the two focal points of a tangent ellipse of $\triangle ABC$. The eccentricity of the tangent ellipse of $\triangle ABC$ is __________ .

10. In $\triangle ABC$, $\angle B < \angle C < 90°$. Point P is inside $\triangle ABC$. Point Q is on AB. $AC=AQ$. Connect PQ. MN vertically bisects PQ and intersects BC at point N. If $BQ=MQ$, the range of $\frac{BN}{CN}$ is __________ .



Section C【Free Response Questions】

(*This section contains 4 questions, each worth 15 points, for a total of 60 points*)

Geometry Part

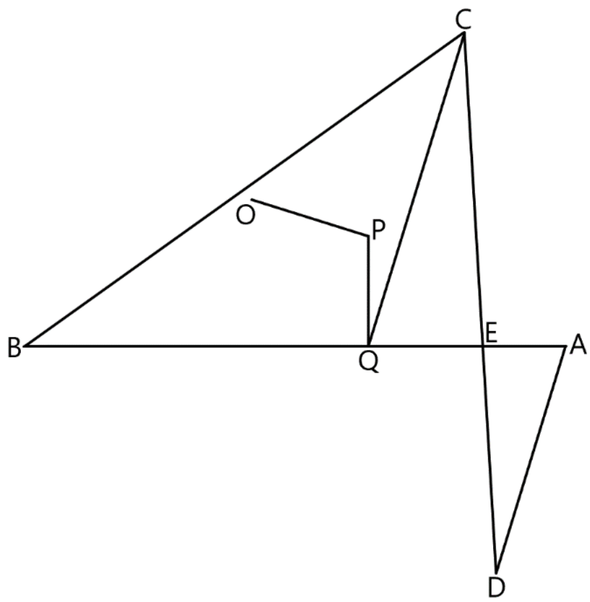
11. $AB = BC = CD$. AB intersects CD at point E. Connect AD.

(**1**) [**3**] If $AE = 1$, $DE = 2$, and $AD = \sqrt{6}$, find BC.

(**2**) [**12**] Let O and P be the circumcenter and incenter of $\triangle BCE$, and connect OP.

① [**5**] Prove: $AD \perp OP$.

② [**7**] CQ is the symmedian of $\triangle BCE$, connect PQ. Prove: $AB \perp PQ$ if and only if $AD /\!/ CQ$. (*Note: The symmedian of a triangle from a certain vertex refers to the line that is the reflection of the line passing through that vertex and the midpoint of the opposite side, over the angle bisector of that vertex*)



Algebra Part

12. *Arithmetical Functions* are functions that define their domain as the set of positive integers. For any two coprime positive integers x, y and arithmetical function $f(n)$, we have $f(xy) = f(x)f(y)$. $f(n)$ is called a *Multiplicative Function*.

(**1**) [**3**] Let $x, k, n \in \mathbb{N}^+$, $x \in \left[2^k, 2^{k+1}\right]$, where $d(n)$ refers to the number of positive factors of n. Prove:

$$\sum_{n \le x} d(n) \le (k+1)\,x$$

(**2**) [**4**] Let $n \in \mathbb{N}^+$, $m = \prod_{i=1}^{t} p_i^{a_i}$ (p is a prime), where $\omega(m) = t$, $\pi(m) = \sum_{i=1}^{t} a_i$, $\lfloor x \rfloor$ represents the largest integer less than or equal to x. Prove:

$$\sum_{m=1}^{n} 5^{\omega(m)} \le \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \cdot d(k)^2 \le \sum_{m=1}^{n} 5^{\pi(m)}$$

(**3**) [**8**] Define that: Given two arithmetical functions $f(n)$, $g(n)$, if there exists a arithmetical function $h(n)$ that satisfies

$$h(n) = \sum_{d|n} f(d) \cdot g(\frac{n}{d})$$

then we call $h(n)$ as the *Dirichlet Convolution* of $f(n)$ and $g(n)$.

① [**3**] If two arithmetical functions $f(n)$, $g(n)$ are both multiplicative functions, and $h(n)$ is their Dirichlet Convolution. Prove that $h(n)$ must also be a multiplicative function.

② [**5**] Let a, m, $n \in \mathbb{N}^+$ where $d(n)$ refers to the number of positive factors of n. Let $f(x) = \sum_{n=1}^{a^m} \frac{d(n)}{n^x}$. If there exists a function $g(x)$ that satisfies $f(x)^2 = \sum_{n=1}^{a^{2m}} \frac{g(n)}{n^x}$, and $g(a^m) = (\mathrm{C}_{n+3}^{3})^3$, find $a_{\min}$.

Number Theory Part

13. Numbers in the form $2^p - 1$ are called *Mersenne numbers*, denoted as M_p.

(**1**) [**2**] Prove the following conclusion:

"p is a prime" is a necessary but not sufficient condition for "M_p is a prime".

(**2**) [**3**] Let p be an odd prime number and let q be a prime factor of M_p. Prove:

$q \equiv 1 \pmod{2p}$

(**3**) [**10**] Determine whether the following statements are true:

Statement 1: For all of the even number t that are greater than 2, there must be a positive integer n that satisfies $n \in [2^{t-1}, 2^t - 1)$ such that

$$n \mid [\gcd(2^t - 1, n) - 1] \cdot (2^{t+1} - 1) + 1$$

Statement 2: For all of the odd number t that are greater than 3, there must be no positive integer n that satisfies $n \in [2^{t-1}, 2^t - 1)$ such that

$$n \mid [\gcd(2^t - 1, n) - 1] \cdot (2^{t+1} - 1) + 1$$

If the statement is true, prove it. If the statement is not true, give a counterexample.

Combinatorics Part

14. Fill the natural numbers 1~9 into the 9 squares of a 3×3 grid, and mark the sum of each row and column. Such a grid is called a *Sudoku Grid*.

（**1**） [**5**] Fill 3 numbers into 3 squares of a Sudoku Grid randomly. Determine the probability that there is still a method that makes the sum of each row and column 15 after filling in the remaining 6 numbers.

（**2**） [**10**] Judging two different Sudoku Grids is based on all positions of the internal 9 numbers. If there are any numbers in different positions, the Sudoku Grids are different. (So there are 9! different Sudoku Grids in total)

① [**5**] List all of the possible positions of all the inner 9 digits that satisfy the sum of each row and column of the right Sudoku Grids.

② [**5**] There are some *Magic Sudoku Grids* that the position of the internal nine numbers is uniquely determined after determining the sum of each row and each column. For example, the left Nine-grid matrix is a Magic Sudoku Grid. Compute the total number of Magic Sudoku Grids.

Sum	12	15	18
6	**1**	**2**	**3**
15	**4**	**5**	**6**
24	**7**	**8**	**9**

Sum	12	16	17
13			
14			
18			